



INTRODUCTION TO THE GEOMETRICAL OBJECTS AND AXIOMS: CONCEPTUAL, DIDACTICAL AND HISTORICAL CONSIDERATIONS

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For a student attending the initial years of the high school, it is not easy to fully realize what a geometrical object is. While speaking, for example, of a triangle, the teacher will underline that its sides have length, but no width and no thickness. However, a pupil has never seen an object of this kind in his daily experience. For, every straight line has a width and a thickness, however minimal they may be. How can we introduce the geometrical objects and, immediately afterwards, the geometrical reasonings so that the learners can accept them not based on a sort of faith act but relying on a real understanding? The best method is to explain their conceptual genesis, also adding some historical elements. Two abstract processes can be identified: the first one gave origin to the abstract objects, the second one to the propositions (axioms) on which the relations of such objects rely. Therefore, we suggest that the teacher dedicates two lessons to introducing the genetic bases of the geometrical thought before dealing with the mathematical details. In what follows, material for the two lessons is supplied.

The most basic geometric concepts, those of point, line, surface and solid arise for practical purposes and have a physical origin.

Suppose you want to measure the length of a stretch of road, perhaps with curves, or the area of a field. How would you proceed? What would you need?

In the case of the road, if you are in a technologically undeveloped civilisation, the first thing that may come to mind is to equip yourself with a straight measuring instrument - on the length of which you and your companions agree - and apply this instrument from the beginning to the end of the road. If the road is not too tortuous, you will be able to measure its length accurately enough.

In the case of the field, things are a bit more complicated. You will have to agree on a surface to be used as a unit of measurement and see how many times this surface fits into the one you are to measure. If the field has a polygonal form, you can 'triangulate' the figure, i.e. divide it into many triangles (see Fig. 1a), and, assuming you know how to calculate the area of a triangle, you can solve your problem. As for figures with curvilinear contours, you can approximate them (which has been done for centuries) to a polygonal shape that reduces the calculation error to within acceptable margins for your purposes.

But let us return to the example of the road: you decide to adopt 10 metres as the unit to measure the road. You therefore build an articulated measuring instrument 10 metres long and start measuring (Figure 1b).

Figure 1a

Example of Triangulation of a Polygonal Figure

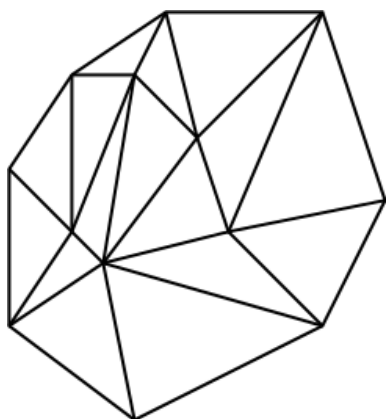


Figure 1b

An Articulated Measuring Instrument



What do you care about your instrument? Does the width matter?

Obviously, your instrument must have a width because there is no such thing as an object that has a length but no width. Furthermore, on a practical level, an instrument that is too thin or one that is too thick would both be uncomfortable, albeit for opposite reasons. However, width in no way enters the measuring process, nor into the result. Only length enters it.

What have you done? You have *abstracted* from the width of the instrument.

In effect, you are creating a new ‘object’, an ‘ideal object’ that only has a length, but no width. You did this, simply because you did not need the width. It is, therefore, easy to understand what it means that, from now on, you will use instruments that, ideally, only have a length.

A new ‘object’, an ‘ideal object’ that only has a length but no width, has been created. This happened because the width played no role in the measuring process. It is, therefore, easy to understand what it means that, from now on, instruments that, ideally, only have a length will be used.

But let's move on: we have built a 10-metre measuring instrument, but who's stopping us from building it longer, perhaps by putting it in a case from which it can be unrolled (Figure 2)? Ideally, we could build instruments as long as we want. Of course, in practice, we have physical limitations: it would be quite inconvenient, for various reasons, to have a measuring instrument 10 km long!

Figure 2

A Measuring Instrument Inserted in a Case



Then we get an idea: why not add to our ‘ideal segment’ another property such that it can be extended indefinitely on both sides? This is very convenient: at least in theory, we would have a measuring instrument that, in one go, tells us immediately how long a certain length is, at least if it is rectilinear.

We then transform our ideal segment into an unlimited straight line. We have thus arrived at the *straight line of geometry* (which means ‘measuring the earth’). An *abstract entity*, but one that is useful for performing concrete operations.

With this, we did not claim to give a precise definition of what a straight line is, but we did get an idea of how such a concept was originally formulated. For the surface as an entity that has a length and a width, but not a thickness, its genesis is analogous: thickness is not necessary if one wants to measure a field or any other surface. Instead, a comparison surface is paramount. It is an ideal surface, namely without thickness. Let us assume, then, that in the abstract world we are constructing, there are also surfaces without thickness.

We now come to something similar, but a little more subtle: you are given a strip of cloth, like, for example, the one in the picture (Figure 3). You are asked to measure it. You unroll it and begin to apply a 1m long instrument. Of course: if the strip is very long and you have no way of marking the subsequent 1m measurements, it is almost impossible for you to measure the strip.

Figure 3

A Very Long Strip of Cloth the Learner Must Measure



One possible way to proceed is to mark, for example, with chalk, successive divisions, each one meter long. What is of interest about these divisions? Perhaps their length, or their width? Absolutely not: it only matters that they indicate a series of “points” of the piece of cloth so that one can say, “This is one meter,” “This is another meter,” and so on.

There arises, then, the idea of considering an entity whose length or width are of no interest, but only its capability to indicate and separate elements within a rectilinear continuum is important. A new geometric entity has been thus constructed. It holds the paradoxical but not contradictory property of having no parts. It is the pure indication of a subdivision operation: it is the geometric point.

In practice, there is nothing analogous to the geometric point, but it is an abstraction that is convenient: its use is convenient. This perhaps explains the origin of the much discussed and criticized Euclidean definition of a point as “that which has no parts.” (Euclid 2008, p. 6).

What has been described can be called the first process of abstraction, that is, one in which abstract objects are constructed that have the described origin and on which certain operations are performed. The Egyptian and Babylonian civilizations had already developed this first process of abstraction.

What I have given here is the idea of an empirical, purpose-related genesis of geometric entities. One can also think of a conceptualist genesis: the idea of a point is reached because, starting from very small spheres, men are able to think of a progressive reduction of such spheres until reaching the abstract idea that there only one unextended point remains. I am, likewise, convinced that without practical necessities, the idea of a dimensionless point would have never been reached.

Finally, an even more idealistic position than conceptualism is that of those who believe that mathematical ideas are innate. This is Platonism.

To summarize: what are the characteristics of geometry analysed so far? A world of abstract objects has been constructed. It is derived from specific necessities of practical life, but the abstract objects have different features in respect to the ordinary ones. Specifically, they lack some ordinary properties because every ordinary object is extended in the three spatial dimensions, whereas the abstract point has no dimension, the abstract line has one, and the abstract surface has two. These new objects, though conceived because of practical reasons, create a geometrical world which is not our perceptive and physical universe. It is an abstract world. For the learners, it might be interesting to know that this level of abstraction had already been obtained by the Egyptians and by the Babylonians.

An example drawn from Egyptian geometry can be useful. Let us consider the problem 56 of the Rhind papyrus, from which it is possible to deduce that probably the Egyptians also knew some trigonometric functions useful in the pyramids' construction.

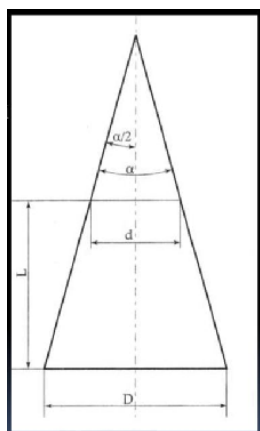
Given a cone (whose section with a vertical plane is shown in Figure 4a), one defines «conicity» C as the ratio between the difference of the diameters D and d of two parallel sections of the cone and the distance L between these two sections if measured axially. Therefore, conicity only depends on the cone's opening angle. It is trivial to show that the identity holds.

$$C = \frac{D-d}{L} = 2 \tan \frac{\alpha}{2}$$

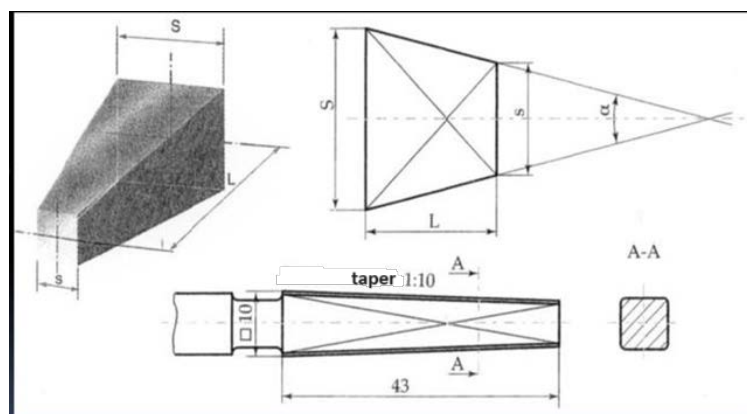
Analogously, for pyramids (in our case we are only interested in those with a square base), «taper» is defined (Figure 4b).

Figure 4a

Exposition of the Concept of Conicity

**Figure 4b**

Exposition of the Concept of Taper



The Egyptians used exactly this concept and called it *seqt*. The *seqt* represents the inward slope of a wall and is simply half the taper.

Problem 56 asks for the *seqt* of a square-based pyramid whose side is 360 cubits and whose height is 250 cubits. To solve the problem, Ahmes (the writer of Rhind papyrus) divides 360 by 2. He then divides the result by 250, so obtaining the tangent of $\alpha/2$, whose value is

expressed as $\frac{1}{2} + \frac{1}{5} + \frac{1}{40}$.

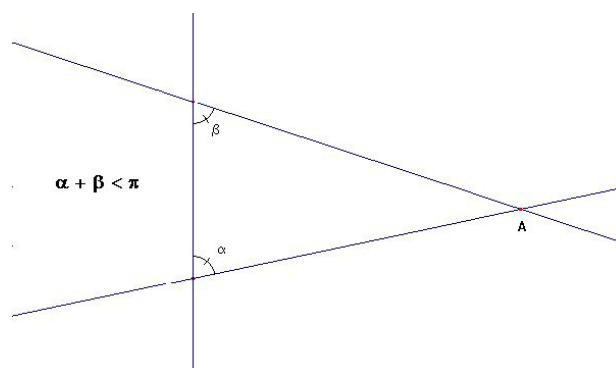
This example shows that the Egyptians were able to deal with abstract geometrical objects. The Egyptians and Babylonians knew how to calculate lengths, areas, and volumes. They already knew some intuitively deduced truths, such as the fact that the area of a rectangle is base by height and the area of a triangle is base by height divided by 2. They were able to combine these simple truths to obtain more complex truths. They even knew some specific cases of the so-called Pythagorean theorem.

However, how have we proceeded so far? Our ordinary intuition has been applied to abstract objects. The foundations of geometric reasoning have not been established, or rather, they have been assumed to be identical to those of ordinary reasoning. Since, however, geometric reasoning concerns abstract entities, it seems illegitimate to apply all and only 'ordinary' reasoning to them. We would need starting points specifically designed for abstract objects. In other terms: it is necessary to establish a set of initial propositions which hold for the objects of the abstract world we have created. Such propositions are not proved, but they are the bases on which all the other properties of the abstract world will be proved. Therefore, there is a *first process of abstraction* through which we construct abstract objects and a *second process of abstraction* through which the basic laws governing such an objectual world are established. For the students to fully understand geometrical thought, it is important that they grasp the nature of the two steps. The merit of the second step is of Euclid (IV-III B.C.) who carried out it in his seminal *Elements*. He stated the following fundamental propositions as axioms or postulates.

1. Let it have been postulated to draw a straight line from any point to any point.
2. And to produce a finite straight line continuously in a straight line.
3. And to draw a circle with any centre and radius.
4. And that all right angles are equal to one another.
5. And that if a straight line falling across two (other) straight lines makes internal angles on the same side (of itself whose sum is) less than two right angles, then the two (other) straight lines, being produced to infinity, meet on that side (of the original straight-line) that the (sum of the internal angles) is less than two right angles (and do not meet on the other side. See Figure 5). (Euclid 2008, p. 7).

Figure 5

A Representation of Euclid's Fifth Postulate, the Famous Axiom of Parallels



The first three axioms state that Euclidean geometry is the geometry of the ruler and compass. No other tools: given two points, it is possible to draw the segment joining them; this segment can be extended as far as one wants (by line, Euclid always means a segment, never an infinite line, unless he specifies otherwise) and one can draw a circle, obviously with a compass. Therefore, only constructions made with a ruler and compass are permitted: a signed ruler or a square cannot be used. Solving a problem, in this context, means solving it with a ruler and compass.

A methodological reflection is useful in teaching: the world of Euclidean geometry has been created. To prove theorems of this universe, one must only use Euclidean tools. Obviously, other choices can be made, for example, allowing the use of richer tools, such as a square, a signed ruler, or an ellipsograph. It will be useful to remind the learners of constructions which cannot be done with a ruler and compass. Examples: 1) duplicating a cube of a given edge, namely, to construct a cube whose volume is double that of a given cube. In itself, the cube is constructible with a ruler and compass, but not if one claims to construct a specific cube that has twice the volume of a given one. 2) Construct a square equivalent to a circle. Separately, a circle and a square are trivially constructible with a ruler and compass, but not if you set the previous specific condition. Therefore, it is useful in teaching geometry to get across the idea that mathematics is a discipline related to the means of construction one decides to use. This aspect is reflected in the postulates. Mathematics is not a kind of ‘machine to prove theorems’, but a careful analysis of the limits of the conceptual tools one decides to use. Of course, one can decide to use all mathematics, but one must be aware that it is, nevertheless, a choice. That is why great mathematicians have often given different demonstrations of the same theorem.

Let us come back to the analysis of postulates: The fourth axiom tells us that the measure of a right angle depends neither on the way in which we arrange it nor on its position in the Euclidean plane. In other words: of the right angle, but of all angles in general, only the measure has geometric significance, not the position, nor the dimensions of the sides.

A further consideration is helpful from an educational perspective: Every system of axioms preserves something. It can also be interpreted as a set of transformation rules with invariants. Without invariants, there would be no mathematics or physics. The most general Euclidean transformations are shape-preserving similarities (relations between segments and angles). The fourth axiom, which claims the invariance of the right angles (and, in fact, of every angle), can be interpreted as the first step towards the theory of similarity and, hence, towards the Euclidean concept of form.

The fifth is the famous postulate of parallels. Euclid, in fact, defines (def. XXIII) as parallel two lines that do not meet each other on either side. Then postulating that if a line falling between two others forms internal angles on the same side (conjugate) less than two right angles (such that the sum is less than two right angles), the two lines, prolonged indefinitely, will meet on the side where the sum of the angles is less than two right angles, is equivalent, by trivial contraposition, to postulating that ‘if two lines do not meet on either side (i.e. they are parallel), then a line falling between them forms conjugate angles equal to two right angles’. Therefore, the fifth postulate expresses a condition of parallelism.

A conceptual division of the postulates into groups could be this: axioms 1-3 define elementary constructions that can be performed by ruler and compass; axiom 4 establishes a uniformity property of the Euclidean plane; and axiom 5 establishes angular relations connected to the incidence of lines. This subdivision is perfectly fine.

However, another is also possible: axioms 1,3,4 concern finite properties of the Euclidean plane, while axioms 2 and 5 regard the behaviour of lines at infinity (prolongation and parallelism). It is a well-known, though very interesting, story that, since antiquity, attempts have been made to prove the fifth postulate as a theorem deducible from the other four because its infinite character and conceptual complexity seemed to collide with the simplicity of the

other four. Proclus (412-485 A.D.) already tried to deduce the fifth postulate from the other four. History has taught us that Euclid, to whom the entire theory of parallels is probably due, was much wiser than his successors, up to Gauss (1777-1855).

Postulate 2 seems, so to speak, innocuous, but it is not, like any other proposition concerning infinity, even in its potential form. Riemann (1826-1866) did, in fact, find a geometry, perfectly consistent with postulates 1,2,4, but in which the line has finite length.

In these considerations, two processes of abstraction have been considered: the former concerns the creation of abstract geometrical objects; the latter, which is more difficult and refined, regards the construction of a language and of a series of basic assumptions that are specific for such objects. It represents the first and fundamental step which progressively led to a formalized language for mathematics. For the students to acquire a satisfactory view of geometry and of the other mathematical abstractions, we consider it crucial that they are introduced to this discipline through two historical-methodological lessons (for the importance of the history of mathematics in an educative context, see, e.g., Bussotti 2012, 2013, 2017, 2021. With regard to the methodological aspects of mathematics education, see e.g., Mudaly & Mpofu 2019, Ersoy & Drağyar 2022, Kurban 2024), whose development has been described here.

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